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On the generalized Legendre transform and monopole metrics

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Abstract: In the generalized Legendre transform construction the Kähler potential is related to a particular function. Here, the form of this function appropriate to the k-monopole metric is calculated from the known twistor theory of monopoles.

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Contents

1. Introduction

Monopole moduli spaces are hyper-Kähler and, therefore, have a twistor description. Such a description is given in [1]. More recently, Ivanov and Roček used the generalized Legendre transform of [16] to construct the metric on the 2-monopole moduli space [15]. The relationship between these two twistor constructions is clarified in this paper.

The twistor space of a hyper-Kähler manifold, M, is a trivial fiber bundle, $Z = M \times \mathbb{P}^1$, with a holomorphic symplectic form ω, which is an $\mathcal{O}(2)$ section over $\mathbb{P}^1$. $\mathbb{P}^1$ is covered by two affine patches, U_0 and U_1. If ζ is the usual projective coordinate on U_0, then ω is given on the fiber over ζ by $\omega = (\omega_2 + i\omega_3) + 2\zeta\omega_1 - \zeta^2(\omega_2 - i\omega_3)$ where ω_1, ω_2 and ω_3 are the covariantly constant 2-forms which hyper-Kählerity implies exist on M.

The generalized Legendre transform construction shows how the Kähler potential may be calculated, if ω is assumed to have a certain form. In subsection 1.1 of this introductory section, there is a brief review of this construction. This is followed, in subsection 1.2, by a brief review of the twistor theory of monopoles and the related twistor theory of monopole moduli spaces. In section 2, this twistor theory is re-expressed as a generalized Legendre transform. The Legendre transform constraints are then the Ercolani-Sinha conditions [7, 12].

1.1 Twistors and the generalized Legendre transform

The generalized Legendre transform construction of [11, 16, 15] concerns twistor spaces with k intermediate holomorphic projections $Z \to \mathcal{O}(2n_j) \to \mathbb{P}^1$, where $j = 1 \ldots k$ and n_j are integers. In the example of interest in this paper $n_j = j$ and for ease of notation attention is restricted to that case. This requirement is equivalent to the existence of k coordinates $\alpha^j(\zeta)$ on Z, so that $\alpha^j(\zeta)$ is a degree $2j$ polynomial:

$$\alpha^j = \sum_{a=1}^{2j} w_a^j \zeta^a , \tag{1.1}$$

satisfying the reality condition $\alpha^j(\zeta) = (-1)^j \zeta^{2j} \overline{\alpha^j(-1/\bar{\zeta})}$.

The construction is derived from the patching formulae relating quantities over U_0 and U_1. The coordinate on U_1 is given by $\tilde{\zeta} = 1/\zeta$. Since ω is an $\mathcal{O}(2)$ line bundle, it is given on U_1 by $\tilde{\omega}$, where

$$\tilde{\omega} = \frac{1}{\zeta^2}\omega \tag{1.2}$$

on $U_0 \cap U_1$. Similarly, the α^j coordinates are related by

$$\tilde{\alpha}^j = \frac{1}{\zeta^{2j}}\alpha^j . \tag{1.3}$$

This means that, if (α^j, ξ^j, ζ) are coordinates for the whole of Z, such that

$$\omega = \sum_{j=1}^{k} d\alpha^j \wedge d\xi^j , \tag{1.4}$$

then, the patching formula for ξ^j must be

$$\tilde{\xi}^j = \zeta^{2j-2}\left(\xi^j + \frac{\partial H}{\partial \alpha^j}\right) \tag{1.5}$$

for some function $H(\alpha^j)$.

The expansion of these coordinates as power series is now considered. The patching formulae will place constraints on the values of certain coefficients in these expansions and these constraints will be unified as constraints on a single function F. By expanding the symplectic form in ζ, it is possible to calculate the Kähler potential for the metric on M in terms of F.

Assuming that ξ^j is non-singular near $\zeta = 0$;

$$\xi^j = \sum_{n=0}^{\infty} x_n^j \zeta^n , \qquad \tilde{\xi}^j = \sum_{n=0}^{\infty} y_n^j \zeta^{-n} . \tag{1.6}$$

Using the residue theorem and the patching formula (1.5), this means that

$$x_m^j = \frac{1}{2\pi i}\oint_0 \xi^j \frac{d\zeta}{\zeta^{m+1}} = \frac{1}{2\pi i}\oint_0 \tilde{\xi}^j \frac{d\zeta}{\zeta^{m+2j-1}} - \frac{1}{2\pi i}\oint_0 \frac{\partial H}{\partial \alpha^j}\frac{d\zeta}{\zeta^{m+1}} , \tag{1.7}$$

where 0 is the small contour surrounding $\zeta = 0$. The integral of $\tilde{\xi}^j$ does not give the coefficient y^j_{2-m-2j}, because the contour around $\zeta = 0$ may enclose branch cuts. It is assumed that the contribution from these cuts can be expressed as an integral of some new function, H', around some contour, c. This integral is the effect of moving the contour of the $\tilde{\xi}^j$ integral from a small contour around zero to a small contour around infinity. This technique is justified by example and is dealt with carefully in the specific case considered below. Thus,

$$x^j_m = y^j_{2-m-2j} - \frac{1}{2\pi i}\oint_c \frac{\partial H'}{\partial \alpha^j}\frac{d\zeta}{\zeta^{m+1}} - \frac{1}{2\pi i}\oint_0 \frac{\partial H}{\partial \alpha^j}\frac{d\zeta}{\zeta^{m+1}}\,. \tag{1.8}$$

A function F is defined as

$$F = -\frac{1}{2\pi i}\oint_c H'\frac{d\zeta}{\zeta^2} - \frac{1}{2\pi i}\oint_0 H\frac{d\zeta}{\zeta^2}\,. \tag{1.9}$$

By the chain rule

$$\frac{\partial F}{\partial w^j_n} = -\frac{1}{2\pi i}\oint_c \frac{\partial H'}{\partial \alpha^j}\zeta^n\frac{d\zeta}{\zeta^2} - \frac{1}{2\pi i}\oint_0 \frac{\partial H}{\partial \alpha^j}\zeta^n\frac{d\zeta}{\zeta^2}\,. \tag{1.10}$$

Therefore, from (1.8)

$$\frac{\partial F}{\partial w^j_0} = x^j_1\,, \qquad \frac{\partial F}{\partial w^j_1} = x^j_0 \tag{1.11}$$

and, for $0 < a < 2j-2$,

$$\frac{\partial F}{\partial w^j_a} = 0\,. \tag{1.12}$$

The symplectic form can be expanded in ζ to determine ω_1 on the fiber above $\zeta = 0$. The coordinates on this fiber are $\alpha^j(0) = w^j_0$ and $\xi^j(0) = x^j_0$. It can then be shown that the Kähler form for the hyper-Kähler manifold, M, is given by

$$K(w^j_0, x^j_0) = F(w^j_0, w^j_1) - \sum_{j=1}^{k} x^j_0 w^j_1 - \sum_{j=1}^{k} \overline{x}^j_0 \overline{w}^j_1\,, \tag{1.13}$$

where w^j_1 is related to x^j_0 by (1.11) [11, 15]. This is the generalized Legendre transform construction.

1.2 Twistors and monopoles

The twistor theory of monopoles is described in [9, 10]. A k-monopole is equivalent to a curve, S, in $T\mathbb{P}^1$ of the form

$$P(\eta,\zeta) = \eta^k + \sum_{j=1}^{k} \alpha^j \zeta^{k-j} = 0\,, \tag{1.14}$$

where α^j is, as before, a degree $2j$ polynomial satisfying the reality condition. η is the usual coordinate on the fiber of $T\mathbb{P}^1 \to \mathbb{P}^1$. S is called the spectral curve. In addition to the reality condition on α^j, it must also satisfy a number of algebraic-geometric conditions. In particular it is required that the L^2 bundle over the spectral curve must be trivial.

The patches U_0 and U_1 on $\mathbb{P}^1$ lift to patches on $T\mathbb{P}^1$. These, in turn, give patches on the spectral curve: these will also be called U_0 and U_1. The triviality of the L^2 bundle over the spectral curve, means that there is a section given by two nowhere vanishing holomorphic functions f_0 on U_0 and f_1 on U_1 satisfying

$$f_0(\eta,\zeta) = e^{-2\eta/\zeta} f_1(\eta,\zeta) \tag{1.15}$$

on the intersection $U_0 \cap U_1$.

In [7], explicit integral conditions are given for the existence of such a section. There must exist a 1-cycle c so that any global holomorphic 1-form, Ω, satisfies

$$\oint_c \Omega = -2\sum_{j=1}^{k} \eta_j(0) g_j \,, \tag{1.16}$$

where $\eta_j(0)$ is the value of η at 0_j, the point on the jth sheet above $\zeta = 0$. g_j is defined by writing $\Omega = g_j d\zeta$ at 0_j. These are the Ercolani-Sinha conditions. c must be primitive if the L^s bundle on S is nontrivial for $0 < s < 2$: another necessary condition.

In the calculation of section 2, the relationship between c and the L^2 section will be important. If $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ is a canonical homology basis,

$$c = \sum_{r=1}^{g} (n_r a_r + m_r b_r) \,, \tag{1.17}$$

where $n_r = \frac{1}{2\pi i}\oint_{b_r} d\log f_1$ and $m_r = -\frac{1}{2\pi i}\oint_{a_r} d\log f_1$ are integers.

The twistor data for monopoles can also be used to derive a twistor space for the k-monopole moduli space, M_k. Coordinates for M_k are provided by the rational map description [6]. This description relates a k-monopole to a degree k based rational map: $p(z)/q(z)$. $q(z)$ is a monic polynomial of degree k and $p(z)$ is a polynomial of degree $k-1$, which has no factors in common with $q(z)$. Following Hurtubise [14], the rational map for a monopole can be constructed from the spectral curve and the trivialization of the L^2 bundle. A direction, ζ, is chosen and

$$\begin{aligned} q(z;\zeta) &= P(z,\zeta) \\ p(z;\zeta) &\equiv f_0(z,\zeta) \bmod q(z) \,. \end{aligned} \tag{1.18}$$

Now, if $q(z)$ has distinct roots, $\eta_1, \ldots, \eta_k$, coordinates for M_k are given by $(\eta_1, \ldots, \eta_k, p(\eta_1), \ldots, p(\eta_k))$. Atiyah and Hitchin point out in [1], that the symplectic form

$$\omega = \sum_{i=1}^{k} \frac{dp(\eta_i)\wedge d\eta_i}{p(\eta_i)} \tag{1.19}$$

has the property that $\tilde{\omega} = \zeta^{-2}\omega$ and is, therefore, an $\mathcal{O}(2)$ section over $\mathbb{P}^1$.

In the next section, the relationship between the rational map and the spectral curve is exploited to clarify the relationship between this symplectic form and the generalized Legendre transformation.

2. The generalized Legendre transform and monopole moduli spaces

The formula for the spectral curve (1.14) defines η_i as roots of a polynomial equation whose coefficients are $\mathcal{O}(2j)$ sections. Because of this, ω can be rewritten

$$\omega = \sum_{i=1}^{k}\sum_{j=1}^{k-1} \frac{dp(\eta_i)}{p(\eta_i)} \wedge \frac{\partial \eta_j}{\partial \alpha^j} d\alpha^j = \sum_{j=1}^{k-1} d\xi^j \wedge d\alpha^j \,, \tag{2.1}$$

where

$$\xi^j = \sum_{i=1}^{k} \frac{\partial \eta_i}{\partial \alpha^j} \chi(\eta_i) \tag{2.2}$$

and $\chi(\eta,\zeta) = \log p(\eta;\zeta)$. ξ^j is not defined on the spectral curve, since χ is not. In general, $\oint_a d\chi \neq 0$ for a nontrivial cycle a. χ can be defined by cutting the surface. The 1-forms $d\chi$ and $d\xi^j$ are defined on the uncut surface.

The patching formula for χ follows from the L^2 patching formula (1.15), since $\tilde{\chi} = \log f_1$,

$$\tilde{\chi} = \chi + \frac{2\eta}{\zeta} \,. \tag{2.3}$$

This, in turn, provides a patching formula for ξ^j,

$$\tilde{\xi}^j = \zeta^{2j-2}\left(\xi^j + 2\sum_{i=1}^{k} \frac{\partial \eta_i}{\partial \alpha^j}\frac{\eta_i}{\zeta}\right) = \zeta^{2j-2}\left(\xi^j + \frac{\partial}{\partial \alpha^j}\sum_{i=1}^{k}\frac{\eta_i^2}{\zeta}\right) . \tag{2.4}$$

Therefore, ξ^j has a Legendre transform patching formula with

$$H = \sum_{i=1}^{k} \frac{\eta_i^2}{\zeta} \,. \tag{2.5}$$

This is the hamiltonian function mentioned in [1].

Now, as before, the integral around $\zeta = 0$ is considered. Because of the particular form of the sum in the expression for H, the integral on $\mathbb{P}^1$ can be written as an integral on the spectral curve

$$\frac{1}{2\pi i}\oint_0 \frac{\partial}{\partial \alpha^j} H \frac{d\zeta}{\zeta^{m+1}} = \frac{1}{2\pi i}\oint_0 \frac{\partial}{\partial \alpha^j}\sum_{i=1}^{k}\frac{\eta_i^2}{\zeta}\frac{d\zeta}{\zeta^{m+1}} = \frac{1}{2\pi i}\oint_{\sum_{i=1}^{k} 0_i} \frac{\partial}{\partial \alpha^j}\frac{\eta^2}{\zeta}\frac{d\zeta}{\zeta^{m+1}} \,, \tag{2.6}$$

where 0_j is the small contour on the jth sheet of the spectral curve around the lift of $\zeta = 0$ to that sheet.

Next, the contour in the $\tilde{\xi}^j$ integral must be moved from 0 to ∞. This is not difficult if the integral is first rewritten as an integral on the spectral curve:

$$\frac{1}{2\pi i}\oint_0 \tilde{\xi}^j \frac{d\zeta}{\zeta^{m+2j-1}} = \frac{1}{2\pi i}\oint_{\sum_{i=1}^{k} 0_i} \frac{\partial \tilde{\eta}}{\partial \tilde{\alpha}^j}\tilde{\chi}\frac{d\zeta}{\zeta^{m+2j-1}} \tag{2.7}$$

$\tilde{\chi}$ is defined on the $4g$-gon, formed by cutting the spectral curve along the canonical homology 1-cycles a_r and b_r for $r = 1, \ldots, g$. Although the spectral curve is obtained from the $4g$-gon by identifying appropriate edges, the function $\tilde{\chi}$ does not respect the identifications. In fact, since, for example,

$$\oint_{b_1} d \log f_1 = 2\pi i n_1 \,, \tag{2.8}$$

the value of $\tilde{\chi}$, at a point on the a_1^{-1} edge, is $2\pi n_1$ larger than its value at the corresponding point on the a_1 edge. This means that

$$\frac{1}{2\pi i} \oint_{\text{edge}} f(\zeta)\tilde{\chi} d\zeta = -\sum_{r=1}^{g} \left(n_r \oint_{a_r} f(\zeta) d\zeta + m_r \oint_{b_r} f(\zeta) d\zeta \right) = -\oint_c f(\zeta) d\zeta \,, \tag{2.9}$$

where $f(\zeta)$ is any function which is well-behaved on the edge and c is the special homology cycle mentioned above (1.17). Furthermore, it is easy to see that

$$\sum_{i=1}^{k} 0_i + \sum_{i=1}^{k} \infty_i = \text{edge} \tag{2.10}$$

and so

$$\begin{aligned} \frac{1}{2\pi i} \oint_0 \tilde{\xi}^j \frac{d\zeta}{\zeta^{m+2j-1}} &= \frac{1}{2\pi i} \oint_\infty \tilde{\xi}^j \tilde{\zeta}^{m+2j-3} d\zeta + \oint_c \frac{\partial \tilde{\eta}}{\partial \tilde{\alpha}^j} \frac{d\zeta}{\zeta^{m+2j-1}} \\ &= \frac{1}{2\pi i} \oint_\infty \tilde{\xi}^j \tilde{\zeta}^{m+2j-3} d\zeta + \oint_c \frac{\partial \eta}{\partial \alpha^j} \frac{d\zeta}{\zeta^{m+1}} \,. \end{aligned} \tag{2.11}$$

Thus, the Legendre transformation of the k-monopole metric is given by

$$F = -\oint_c \frac{\eta}{\zeta^2} d\zeta - \frac{1}{2\pi i} \oint_{\sum_{i=1}^{k} 0_i} \frac{\eta^2}{\zeta^3} d\zeta \,. \tag{2.12}$$

This F has also been considered by Roger Bielawski [3].

F is composed of integrals on the spectral curve, rather that on $\mathbb{P}^1$ itself. The integrals can be rewritten as integrals on $\mathbb{P}^1$, although they become less succinct. If the integral in the $k = 2$ case is rewritten in this way the F used by Ivanov and Roček to calculate the Atiyah-Hitchin metric [15] is recovered. In that case the two branches over $\zeta = 0$ differ only by a sign in η and the special contour, c, is an equator.

In the $k = 2$ case, the constraint (1.12) arising in the generalized Legendre transform construction is precisely the one that Hurtubise demonstrates must be satisfied for a spectral curve to ensure triviality of the L^2 bundle [13]. In fact, it is true for all k, that the generalized Legendre transformation constraints are the L^2 triviality conditions (1.16).

This is demonstrated by using the spectral curve equation to rewrite the integrands in the constraint equations.

$$\frac{d}{dw_a^j} P(\eta, \zeta) = \frac{\partial P}{\partial \eta} \frac{\partial \eta}{\partial w_a^j} + \frac{\partial P}{\partial w_a^j} = 0 \tag{2.13}$$

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implies that

$$\frac{\partial \eta}{\partial w_a^j} = -\frac{\zeta^a \eta^{k-j}}{\partial P/\partial \eta}\,. \tag{2.14}$$

Therefore, the constraint equation requires that

$$-\frac{1}{2\pi i}\oint_{\sum_{i=1}^{k} 0_i} \frac{2\eta^{k-j+1}\zeta^{a-2}}{\partial P/\partial \eta}\frac{d\zeta}{\zeta} = \oint_c \frac{\eta^{k-j}\zeta^{a-2}d\zeta}{\partial P/\partial \eta}\,, \tag{2.15}$$

where $2 \leq a \leq 2j-2$, or, put another way

$$\oint_c \Omega^{ja} = -\frac{1}{2\pi i}\sum_i \oint_{0_i} \frac{2\eta\Omega^{ja}}{\zeta}\,, \tag{2.16}$$

where

$$\Omega^{ja} = \frac{\eta^{k-j}\zeta^{a-2}d\zeta}{\partial P/\partial \eta} \tag{2.17}$$

is a global holomorphic 1-form. In fact, these 1-forms form a basis for the global holomorphic 1-forms on the spectral curve. There, using the residue theorem on the right-hand side of the equations shows that it is the Ercolani-Sinha constraint (1.16). It may be noted that the right hand side of this equation is actually zero for $j \neq 1$ [12].

Thus, the generalized Legendre transform for monopole moduli space can be derived from the twistor description of monopoles. The constraints on F are the Ercolani-Sinha constraints. However, it should be emphasized that the Ercolani-Sinha constraints only ensure that the L^2 bundle is trivial and that the L^s bundle is not trivial if $s < 2$. These are necessary conditions. They are not sufficient. The additional condition requires that $H^0(S, L^s(k-2)) = 0$ for $0 < s < 2$. It is possible that this condition may also be interpreted in terms of the generalized Legendre transformation.

The crucial step in calculating the constraints on F, is the derivation of the coefficient form of the patching equation by expanding the patching formula and moving the contour in the $\tilde{\xi}^j$ integral. Mimicking this derivation also clarifies the relationship, explained in [12], between the Ercolani-Sinha and Corrigan-Goddard conditions [5].

2.1 The point dyon limit

In this paper F has been calculated from the known k-monopole symplectic form. In [15] Ivanov and Roček found F for the 2-monopole metric by making a guess based on the known asymptotic metric and then verifying this guess by explicit calculation of the Kähler form. In fact, the asymptotic metric has been calculated for k monopoles by examining the dynamics of point dyons [8], this approximation was confirmed in [2]. It is a simple matter to derive this asymptotic metric from F, thereby reversing, for general k, the original derivation of F for $k = 2$.

The spectral curve for a single monopole located at $(\mathrm{Re}\, z, \mathrm{Im}\, z, x)$ is

$$\eta - z + 2x\zeta + \bar{z}\zeta^2 = 0\,. \tag{2.18}$$

This is the sphere in $T\mathbb{P}^1$ corresponding to all the lines through the monopole location. The spectral curve for k-monopoles located at well separated points $(\mathrm{Re}\, z_i, \mathrm{Im}\, z_i, x_i)$ is approximated with exponential accuracy by the product of spheres [2]

$$\prod_{i=1}^{k} (\eta - \gamma_i) = 0\,, \tag{2.19}$$

where

$$\gamma_i = z_i - 2x_i\zeta - \bar{z}_i\zeta^2\,. \tag{2.20}$$

The i and j spheres touch at two points, the two roots of $\gamma_i = \gamma_j$:

$$\zeta_{ij}^{\pm} = \frac{x_{ij} \pm \sqrt{x_{ij}^2 + |z_{ij}|^2}}{\bar{z}_{ij}} \tag{2.21}$$

where $z_{ij} = z_i - z_j$ and $x_{ij} = x_i - x_j$. It is known that the special contour c must change sign under the reality transformation $\zeta \to -1/\bar{\zeta}$ [12] and so must be a sum of contours which run from ζ_{ij}^- to $\zeta_{ij}^+ = \zeta_{ji}^-$ on sphere i and then from ζ_{ji}^- back to $\zeta_{ji}^+ = \zeta_{ij}^-$ on sphere j. In order for F to generate the asymptotic metric c must be a sum of all such contours.

F can be rewritten in terms of integrals on $\mathbb{P}^1$. It is

$$F = -\sum_{i\neq j} \int_{ij} \frac{\gamma_{ij}}{\zeta^2} d\zeta - \sum_i \frac{1}{2\pi i} \oint_0 \frac{\gamma_i^2}{\zeta^3} d\zeta\,, \tag{2.22}$$

where $\gamma_{ij} = \gamma_i - \gamma_j$ and the ij integral is along the line running from ζ_{ij}^- to ζ_{ij}^+. In order to change the line integrals into contour integrals a $\log \gamma_{ij}$ is introduced into the integrand, thus,

$$F = -\sum_{i\neq j} \frac{1}{2\pi i} \oint_{ij} \frac{\gamma_{ij} \log \gamma_{ij}}{\zeta^2} d\zeta - \sum_i \frac{1}{2\pi i} \oint_0 \frac{\gamma_i^2}{\zeta^3} d\zeta\,, \tag{2.23}$$

where the ij integral is now around the figure of eight contour enclosing the two zeros of γ_{ij}. This F has been discussed in [4] and gives the correct asymptotic metric.

3. Conclusions

The function F appropriate to the generalized Legendre transform construction of multimonopole metrics is calculated by a simple change of variables. This function is a contour integral over the spectral curve. The constraints on F are precisely the integral constraints on the spectral curve required to ensure the existence of a trivial L^2 bundle. In practice these constraints are difficult to apply.

The generalized Legendre transform was originally derived from a duality transformation on an $N = 4$ supersymmetric σ-model. It would be interesting to understand what relationship this σ-model has to monopoles.

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